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The Free Partially Commutative Lie Algebra: Bases and Ranks

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In this paper, we study the free partially commutative Lie K-algebra $L(A, \theta)$ defined by a commutation relation θ on an alphabet A. Its behavior is very similar to that of the free Lie algebra. Indeed, we obtain in particular a partially commutative version of Lazard's elimination process which allows us to prove that the K-module $L(A, \theta)$ is free and to construct explicitly K-bases for it. We show also how the classical Witt's calculus can be extended to $L(A, \theta)$.

INTRODUCTION

The *free partially commutative monoid* was introduced by P. Cartier and D. Foata in 1969 [CaFo] for the study of combinatorial problems in connection with word rearrangements. Since that time, this monoid has been the subject of many studies. They were principally motivated by the fact that the free partially commutative monoid is a model for concurrent computing. Indeed, the independance or the simultaneity of two actions can be interpreted by the commutation of two letters a and b that code them (see [Mz] for instance).

The rational languages can be studied from the viewpoint of words and of automata or from the formal series viewpoint. In the same way, the study of the free partially commutative monoid has grown in two directions. First, there is a current which is close to usual language theory and which tries to investigate how the classical notions can be extended to languages of partially commutative words, then called *trace languages* (see [Ch, CoPe, Db.1, Db.2, Mt, Oc], for example). In this direction, let us recall the fundamental theorem of W. Zielonka (see [Zi] or [Pe]) that

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characterizes the recognizable traces by introducing the notion of asynchronous automaton. On the other hand, a second research direction took shape around some algebraic structures related to the free partially commutative monoid. For instance, let us cite the studies of the free partially commutative group [Db.1] and of the free partially commutative associative algebra (cf. [Du.1, Th, Sm]).

Let us also point out that several results are true in both the commutative and non-commutative cases. Thus it is very interesting to explore what are the theorems that pass through all the commutation lattice in order to obtain general statements. In this way, we can give for instance the three following recent results:

— The free partially commutative monoid embeds always in the free partially commutative group (see [Db.1]).

— The algebra of partially commutative polynomials on a semiring K is an integral domain (resp. left or right regular) if and only if K is also an integral domain (resp. left or right regular) (see [Du.1]).

— The free partially commutative associative algebra can be naturally equipped with a Hopf algebra structure with antipode leading to formulas related to the notion of partially commutative subword and shuffle (see [Sm]).

In this direction also goes our paper: indeed, it is devoted to the study of the free partially commutative Lie algebra introduced in [Du.1]. We show how to obtain bases of the module underlying this algebra and how to realize some computations of ranks. Let us recall that the free partially commutative Lie algebra can be defined by the presentation (cf. [Bo.2])

$$L(A, \theta) = \langle A; [a, b], (a, b) \in \theta \rangle.$$

In this paper, we show that it is possible to obtain information on a Lie algebra defined by homogeneous relators of the above kind. Thus, our work is to be considered as a contribution to the "combinatorial Lie algebra theory" — that is to say, a study of Lie algebras given by generators and relators similar in its spirit to the classical combinatorial group theory (see [MKS] or [LySc]) — continuing the pioneer works of M. P. Schützenberger and of G. Viennot (see [Sc.1, Sc.2, Vi.1, Vi.2]).

Finally let us end with the structure of this paper. Section I is devoted to the definition of the free partially commutative Lie algebra and to the study of its enveloping algebra. In Section II, we prove a partially commutative version of Lazard's elimination theorem (see [Lz.2] or [Bo.2]) that allows us to prove by a constructive method that the module $L(A, \theta)$ is a direct sum of free Lie algebras. This method permits us without any difficulty to construct bases for $L(A, \theta)$ with the classical methods known for the free Lie algebra (cf. [Vi.2]). Finally, we devoted Section III to the computation of the ranks of the homogeneous components of $L(A, \theta)$ for the multidegree and the total degree. Owing to the fact that the enveloping algebra of $L(A, \theta)$ is the algebra of the corresponding partially commutative polynomials, we show how the classical Witt formulas (see [Bo.2]) can be generalized, hence obtaining new combinatorial numbers.

0. Preliminaries

(1) *The Free Partially Commutative Monoid*

In all this paper, A denotes an alphabet and θ is a *partial commutation relation* on A, that is to say a symmetric subset of $A \times A$[1] which does not contain any element of the diagonal $\Delta_A = \{(a, a), a \in A\}$ of $A \times A$. We often represent θ by its commutation graph (see [Ch]). We call *alphabet of* θ and we denote $\mathrm{Alph}(\theta)$ the subset of A defined by

$$\mathrm{Alph}(\theta) = \{y, \exists z \in A, (y, z) \in \theta\}.$$

We say that a subalphabet B of A is *totally commutative* (resp. *totally non commutative*) for θ iff we have

$$\forall b, c \in B, b \neq c \Rightarrow (b, c) \in \theta \text{ (resp. } (b, c) \notin \theta).$$

Then we denote $M(A, \theta)$ the *free partially commutative monoid* on A that is associated to θ. Let us recall that it is defined by

$$M(A, \theta) = A^*/\equiv_\theta, \qquad (\mathcal{FPCM})$$

where $\equiv_\theta$ denotes the finest monoid congruence of A^* such that

$$\forall (a, b) \in \theta, \qquad ab \equiv_\theta ba.$$

We denote by $\bar{w}$ the image of every word w of A^* by the projection of A^* onto $M(A, \theta)$ defined by $(\mathcal{FPCM})$, i.e., the equivalence class of w for $\equiv_\theta$.

We refer to [Be] or to [Ha] for the notions of graph theory we need in the sequel. Nevertheless let us recall that a *clique* in a graph $G = (X, U)$ is a complete subgraph $C = (Y, V)$ of G, i.e., such that

$$\forall y, z \in Y, \qquad (y, z) \in V.$$

Thus the cliques of the graph of a partial commutation θ correspond to the totally commutative parts of A for θ. The notation $\mathcal{C}\ell(\theta)$ will denote here the set of the non empty cliques of $\theta \cup \Delta_A$.

[1] I.e., such that $(a, b) \in \theta \Leftrightarrow (b, a) \in \theta$.

We need here several notions of degrees on the words of $M(A, \theta)$. Let us begin to introduce them on usual words:

DEFINITION 0.1. Let A be an alphabet. Then we define for every $w \in A^*$ and if necessary for every $a \in A$ the following notions:

(i) The *partial degree* of w in a, denoted $|w|_a$, is the number of a in w.

(ii) The *total degree* of w is the length of w that we shall denote $|w|$.

(iii) The *multidegree* of w is the A-tuple $|w|_A = (|w|_a)_{a \in A} \in \mathbb{N}^{(A)}$.

Observe that the congruence $\equiv_\theta$ is compatible with all the previous notions. Indeed, we can easily check that for every $u, v \in A^*$,

$$u \equiv_\theta v \Rightarrow |u| = |v|, \ |u|_a = |v|_a, \text{ and } |u|_A = |v|_A.$$

Hence this property permits us to speak of the partial degree, of the total degree, or of the multidegree of a word $\bar{w} \in M(A, \theta)$: one need only define it as the corresponding degree of the word w of A^*. This allows us also to use the notations of Definition 0.1 for the words of $M(A, \theta)$. Then we can define the following homogeneous components of $M(A, \theta)$ for every multidegree α in $\mathbb{N}^{(A)}$ and every total degree n in $\mathbb{N}$:

$$M_\alpha(A, \theta) = \{w \in M(A, \theta), |w|_A = \alpha\}$$

and

$$M_n(A, \theta) = \{w \in M(A, \theta), |w| = n\}.$$

Notes. (1) We refer to [Bo.2] for the classical notations on multidegrees we use in Section III of this paper.

(2) For every letter $a \in A$, we denote by ε_a the element of $\mathbb{N}^{(A)}$ all of whose entries are 0 except the ath which is 1.

(2) *The Free Lie Algebra*

We suppose in all this paper that K is a fixed *commutative ring* not reduced to 0. The term "algebra" always means K-algebra. We refer also to [Vi.2] or to [Bo.2] for the definition of the free Lie K-algebra $L(A)$ constructed on the alphabet A and to [Bo.1] for the elementary properties of a Lie algebra.

With the universal property of the free Lie algebra (see [Bo.2]), we can define the following notion that will be of frequent use in this paper:

DEFINITION 0.2 [Bo.2]. Let $\mathcal{G}$ be a Lie K-algebra and let T be a subset of $\mathcal{G}$. Let us denote i (resp. i_T) the natural injection of T into $\mathcal{G}$ (resp. $L(T)$)

and let $j(\mathcal{G}, T)$ be the unique Lie K-algebra morphism such that the following diagram becomes commutative:

$$\begin{tikzcd} T \arrow[r, "i"] \arrow[d, "i_T"'] & \mathcal{G} \\ L(T) \arrow[ur, "j(\mathcal{G}, T)"'] & \end{tikzcd}$$

Then, the subset $T \subset \mathcal{G}$ is said to be *basic* if and only if the Lie morphism $j(\mathcal{G}, T)$ is injective: in this case, the Lie subalgebra $j(\mathcal{G}, T)(L(T))$ is called an embedded free Lie subalgebra of $\mathcal{G}$.

Note. In the sequel, when T is a basic family of a Lie algebra $\mathcal{G}$, there will never be any danger of confusion: hence the Lie subalgebra $j(\mathcal{G}, T)(L(T))$ of $\mathcal{G}$ is denoted by $L(T)$.

We also use in the whole paper the following convention: for all subsets U, V of $L(A)$, $[U, V]$ denotes the set

$$[U, V] = \{[u, v],\ u \in U,\ v \in V\}$$

and not the K-module generated by the above family as it is usually defined.

Let $L(A)$ the free Lie algebra on A. Then, for every word $w \in A^*$ and every element z of $L(A)$, we define $\operatorname{ad} w.z$ by the induction relations

$$\operatorname{ad} 1.z = z$$

$$\text{If } w = xu \text{ with } x \in A \text{ and } u \in A^*,\ \operatorname{ad} w.z = [x, \operatorname{ad} u.z].$$

In the same way, we inductively define the families $(\mathcal{A}_n)_{n \geqslant 1}$ of elements of the free Lie algebra by

$$\mathcal{A}_1 = A \qquad \text{and} \qquad \forall n \geqslant 2,\ \mathcal{A}_n = \bigcup_{p+q=n} [\mathcal{A}_p, \mathcal{A}_q] - \{0\}.$$

For every $n \geqslant 1$, an element of $\mathcal{A}_n$ is called an *n-fold Lie monomial* or a homogeneous *Lie monomial* of degree n. It is easy to show that the family of the Lie monomials of arbitrary degree is a generating family for the K-module $L(A)$ (cf. [Bo.2] or [Ja]).

Let $L(A)$ be the free Lie algebra on the alphabet A and let $\mathcal{A}$ be a subset of $L(A)$. Then we shall denote $\langle \mathcal{A} \rangle_{L(A)}$ the Lie ideal of $L(A)$ generated by $\mathcal{A}$.

The following result will be essential in order to prove that the K-module $L(A, \theta)$ is free: it is *Lazard's elimination theorem* [Bo.2]. Observe that only a very reduced from of this theorem, corresponding to the case $|B| = 1$, was in fact used to construct bases of $L(A)$ (see [Vi.1, Vi.2] or [Bo.2]).

THEOREM 0.1. *Let A be an alphabet and let $B \subset A$. Then the family*

$$T = \{\mathrm{ad}\, w.z,\ w \in B^*,\ z \in A - B\}$$

is a basic family of the Lie ideal $\mathscr{T}$ generated by $A - B$. More precisely, the Lie algebra morphism φ from $L(T)$ into $L(A)$ defined by

$$\forall w \in B^*,\ \forall z \in A - B, \qquad \varphi(\mathrm{ad}\, w.z) = \mathrm{ad}\, w.z$$

is a bijection from $L(T)$ onto $\mathscr{T}$. Moreover, if $\mathscr{T}$ and $L(T)$ are identified, the K-module $L(A)$ admits the following decomposition in direct sum:

$$L(A) = L(B) \oplus L(T).$$

Note. Thus, under the hypotheses of Lazard's elimination theorem, we have

$$[L(T), L(A)] \subset L(T).$$

For every alphabet A (not necessarily finite) totally ordered by $<$, we denote by $\mathscr{L}y_A$ the family of Lyndon words on A (see [Vi.2], [Lo], or [BePe]). For every word f of $\mathscr{L}y_A$, there exists a decomposition,

$$f = gh \qquad \text{with} \quad g \in \mathscr{L}y_A,\ h \in A^+,$$

where g is of maximal length. It can be shown (see [Vi.2]) that $h \in \mathscr{L}y_A$. This permits us to define a bracketing mapping π from $\mathscr{L}y_A$ into $L(A)$ by

$$\forall a \in A, \qquad \pi(a) = a \text{ and for every } f = gh \in \mathscr{L}y_A \text{ as above,}$$
$$\pi(f) = [\pi(g), \pi(h)]$$

It can be proved (cf. [Vi.2]) that $\pi(\mathscr{L}y_A)$ is a basis of the free K-module $L(A)$ called the *Lyndon basis.* It is obviously not the only way to obtain a basis for $L(A)$, but this basis has remarkable properties that are very convenient for practical uses (cf. [Vi.2]).

I. THE FREE PARTIALLY COMMUTATIVE LIE ALGEBRA

(1) DEFINITION. We denote by I_θ the Lie ideal of $L(A)$ defined by

$$I_\theta = \langle [a, b],\ (a, b) \in \theta \rangle.$$

Then we can give the following natural definition:

DEFINITION I.1. Let A be an alphabet and let θ be a relation of partial commutation on A. Then we call *free partially commutative Lie algebra* on A the Lie algebra denoted $L(A, \theta)$ and defined by

$$L(A, \theta) = L(A)/I_\theta.$$

Remark. The free partially commutative Lie algebra is of course the solution of a universal problem similar to the classical one for the free Lie algebra [Du.2]. Indeed, if A is an alphabet equipped with a partial commutation θ, if $\mathscr{G}$ is a Lie K-algebra, and if f is a mapping from A into $\mathscr{G}$ such that

$$\forall (a, b) \in \theta,\ [f(a), f(b)] = 0,$$

then there exists a unique Lie algebra morphism $\bar{f}$ from $L(A, \theta)$ into $\mathscr{G}$ that extends f, i.e., such that the following diagram is commutative (where i_A is the natural injection from A into $L(A, \theta)$):

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_A"'] & \mathscr{G} \\ L(A, \theta) \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Notation. Let $p\ell_\theta$ denote the canonical projection of $L(A)$ on $L(A, \theta)$. Then we can define the homogeneous components of $L(A, \theta)$ from the corresponding homogeneous components of $L(A)$ (cf. [Bo.2]) by

$$L_\alpha(A, \theta) = p\ell_\theta(L_\alpha(A)) \qquad \text{and} \qquad L_n(A, \theta) = p\ell_\theta(L_n(A))$$

for every multidegree $\alpha \in \mathbb{N}^{(A)}$ and every total degree $n \in \mathbb{N}$.

Remark. Since I_θ is generated by homogeneous elements, $L(A, \theta)$ inherits by quotient of the graduations of $L(A)$ for the total degree and the multidegree (cf. [Bo.2]). Thus we can write

$$L(A, \theta) = \bigoplus_{\alpha \in \mathbb{N}^{(A)}} L_\alpha(A, \theta) = \bigoplus_{n \in \mathbb{N}} L_n(A, \theta).$$

(2) *The Enveloping Algebra of* $L(A, \theta)$

We study in this section the enveloping algebra of $L(A, \theta)$. First, let us recall the following definition (cf. [Du.1, Du.2]):

Definition I.2. Let A be an alphabet and let θ be a partial commutation relation on A. Then we call *free partially commutative K-algebra* and we denote $K\langle A, \theta\rangle$ the K-algebra of the monoid $M(A, \theta)$ (cf. [Bo.3]).

Notation. We define the homogeneous components of $K\langle A, \theta\rangle$ by

$$K_\alpha\langle A, \theta\rangle = \bigoplus_{m \in M_\alpha(A,\theta)} K.m \quad \text{and} \quad K_n\langle A, \theta\rangle = \bigoplus_{m \in M_n(A,\theta)} K.m$$

for every multidegree $\alpha \in \mathbb{N}^{(A)}$ and every total degree n.

Let us introduce now J_θ, the *two-sided ideal* of $K\langle A\rangle$, which is generated by the following commutators:

$$J_\theta = \langle ab - ba,\ (a, b) \in \theta\rangle.$$

Then the following result shows that the free partially commutative algebra is the quotient of the free associative algebra by J_θ:

PROPOSITION I.1. *Let us take the previous notations. Then the two following K-algebras are isomorphic*:

$$K\langle A, \theta\rangle \simeq K\langle A\rangle / J_\theta.$$

Proof. Let us consider the unique K-module morphism φ from $K\langle A\rangle$ into $K\langle A, \theta\rangle$ defined on the words of A^* by the relation

$$\forall w \in A^*, \qquad \varphi(w) = \bar{w}.$$

Since φ becomes clearly a K-algebra morphism, it suffices for proving the proposition to show that we have

$$\operatorname{Ker} \varphi = J_\theta. \tag{1}$$

Since the inclusion $J_\theta \subset \operatorname{Ker} \varphi$ is obvious, it suffices to prove that every element of $\operatorname{Ker} \varphi$ is in J_θ to obtain (1). For every $P \in \operatorname{Ker} \varphi$, we have

$$P = \sum_{w \in A^*} p_w w \qquad \text{with} \qquad \sum_{w \in A^*} p_w \bar{w} = 0. \tag{2}$$

Let $(w_i)_{i=1,n}$ denote a set of elements in the support of P which is a section of the equivalence classes for $\equiv_\theta$ of the elements in the support of P. Then we can write

$$P = \sum_{i=1}^{n} \left(\sum_{w \equiv w_i} p_w w \right). \tag{3}$$

Then, according to (2), we have for every i in $[\![1, n]\!]$

$$\sum_{w \equiv w_i} p_w w = 0 \Leftrightarrow p_{w_i} = - \sum_{w \equiv w_i,\, w \neq w_i} p_w.$$

Let us fix now $i \in [\![1, n]\!]$ and let us work only with the elements of the support of P that are equivalent to w_i for the congruence $\equiv_\theta$. Then we have

$$\sum_{w \equiv w_i} p_w w = \sum_{w \neq w_i} p_w w - \left(\sum_{w \neq w_i} p_w \right) w_i = \sum_{w \neq w_i} p_w (w - w_i). \tag{4}$$

But, for every $w \equiv_\theta w_i$, there exists a sequence $(u_j)_{j=1,m}$ of elements of A^* such that the relations hold (cf. [Ch]):

$$u_0 = w, \qquad u_m = w_i$$

$$\forall j \in [\![0, m-1]\!], \qquad u_j = x_j aby_j \text{ and } u_{j+1} = x_j bay_j \text{ with } (a, b) \in \theta.$$

Hence, for every i, we deduce that

$$w - w_i = u_0 - u_m = \sum_{j=0}^{m-1} (u_j - u_{j+1}) = \sum_{j=0}^{m-1} x_j (ab - ba) y_j \in J_\theta.$$

Thus, according to (4) and (3), our computation shows that P belongs to J_θ. This is exactly that we wanted to prove. ▮

Then we obtain again a result of [Du.2] which gives the enveloping algebra of the free partially commutative Lie algebra. It generalizes the classical corresponding theorem for the free Lie algebra (cf. [Bo.2]).

COROLLARY I.2. *The enveloping algebra of $L(A, \theta)$ is $K\langle A, \theta\rangle$.*

Proof. This is an immediate consequence from Proposition 3 and from Corollary 5 of Chap. 2 of [Bo.1], from Proposition I.1 and from the fact that $L(A)$ is a free K-module (see [Vi.2] or [Bo.2]). ▮

II. A BASIS OF $L(A, \theta)$

(1) *Preliminaries*

Let X be an alphabet not necessarily finite. Then let us consider a subset Y of X and a family $\mathrm{Dif} = (z_i^1 - z_i^2)_{i \in I}$, where z_i^1, z_i^2 are letters of X for every i in I. We study the ideal I of $L(X)$ defined by

$$I = \langle Y, \mathrm{Dif} \rangle_{L(X)}$$

since in the next section we reduce the study of a K-basis for $L(X, \theta)$ to those of a K-basis for a Lie algebra of the kind $L(X)/I$. For our study, it will be useful to introduce the equivalence relation $\equiv$ on X defined by

$$x \equiv x' \Leftrightarrow \exists i_1, ..., i_n \in I, \qquad \begin{cases} x = z_{i1} \text{ and } x' = z_{in} \\ \forall j \in [\![1, n-1]\!],\ \pm(z_{ij} - z_{ij+1}) \in \mathrm{Dif}. \end{cases}$$

It is clear that $\equiv$ is really an equivalence relation since $x \equiv x'$ iff there exists a chain from x to x' in the graph constructed on X where we put an edge between two vertices z, z' iff $z - z' \in \mathrm{Dif}$ or $z' - z \in \mathrm{Dif}$. For every $x \in X$, let us denote $C(x)$ the equivalence class of x for $\equiv$. We introduce finally

$$\mathcal{R} = (z_i)_{i \in R},$$

a section of the equivalence classes for $\equiv$ which contain a letter z occurring in Dif (i.e., such that there exists z' with $\pm(z - z') \in \mathrm{Dif}$). Then we can give the result:

PROPOSITION II.1. *With the previous notations, we have*

$$I = \langle Y, (z_i - z)_{i \in R,\, z \in C(z_i)} \rangle.$$

Proof. In order to simplify our proof, let us set

$$J = \langle Y, (z_i - z)_{i \in R,\, z \in C(z_i)} \rangle.$$

Thus we shall show that $I = J$. First, let i be in R and z in $C(z_i)$. Hence, by definition of $\equiv$, there exists a sequence $(x_j)_{j=1,n}$ of letters of X and a sequence $(\varepsilon_j)_{j=1,n}$ in $\{-1, 1\}$ such that

$$\forall j \in [\![1, n-1]\!], \qquad \varepsilon_j(x_j - x_{j+1}) \in \mathrm{Dif} \text{ and } z_i = x_1,\ z = x_n$$

We deduce from this that

$$z_i - z = x_1 - x_n = \sum_{j=1}^{n-1} (x_j - x_{j+1}) \in I.$$

Therefore it follows immediately that $J \subset I$. Conversely, it is clear that for every $i \in I$, $z_i^1 \equiv z_i^2$. Then let us denote z the representative in the family $\mathcal{R}$ of the equivalence class $C(z_i^1) = C(z_i^2)$. Then, we can write

$$z_i^1 - z_i^2 = (z_i^1 - z) + (z - z_i^2) \in J.$$

This implies easily that $I \subset J$ and finally that the equality $I = J$ holds. ∎

We end this preliminary study by a last reduction of the form of I. Thus, let us introduce the subset S of R defined by

$$S=\{i\in R,\ C(z_i)\cap Y\neq\varnothing\}.$$

Then we can easily establish

COROLLARY I.2. *With the previous notations, we have*

$$I=\left\langle Y\cup\bigcup_{i\in S}C(z_i),\ (z_i-z)_{i\in R-S,\ z\in C(z_i)-\{z_i\}}\right\rangle. \qquad (gi)$$

Proof. This is an easy consequence of the previous proposition that we let the reader verify. ∎

Note that with this last writing we do have

$$\left(Y\cup\bigcup_{i\in S}C(z_i)\right)\cap\left(\bigcup_{i\in R-S}C(z_i)\right)=\varnothing. \qquad (iv)$$

This approach and simplification work being done, we can now give our main result for this section:

PROPOSITION II.3. *With the previous notations, let us set*

$$Z=\left(X-\left(Y\cup\bigcup_{i\in R}C(z_i)\right)\right)\cup\bigcup_{i\in R-S}\{z_i\}.$$

Then, the K-module $L(X)$ is a direct sum of I and of the free Lie subalgebra $L(Z)$ of $L(X)$ having Z as basic subset:

$$L(X)=I\oplus L(Z).$$

Proof. First, observe that Z is in fact equal to

$$Z=X-\left(Y\cup\bigcup_{i\in S}C(z_i)\cup\bigcup_{i\in R-S}C(z_i)-\{z_i\}\right).$$

Since Z is a subalphabet of X, it follows that it is really a basic family that generates a free Lie subalgebra of $L(X)$. We now prove that $L(X)$ is the sum of I and of $L(Z)$, i.e., that we have

$$L(X)=I+L(Z). \qquad (1)$$

To prove (1), we show by induction on the degree $\mathscr{d}$ of a Lie monomial α of $L(A)$ that we always have the relation

$$\alpha \in I + L(Z). \tag{2}$$

First, let us suppose that $\mathscr{d} = 1$. Then, several cases can occur: either α is a letter of Y or of $C(z_i)$ for some $i \in S$ and then $\alpha \in I$ according to II.2; or α is a letter of $C(z_i) - \{z_i\}$ for $i \in R - S$, in which case we can write

$$\alpha = (\alpha - z_i) + z_i \in I + Z \subset I + L(Z)$$

by definition of Z and by Corollary II.2; or, finally, $\alpha \in Z$ and then we can at once conclude that $\alpha \in L(Z)$. Hence this ends proving (2) when $\mathscr{d} = 1$. Let us suppose now that the relation (2) is proved for every degree $< \mathscr{d}$ and let α be a Lie monomial of $L(A)$ of degree $\mathscr{d}$. Then, we can write

$$\alpha = [u, v],$$

where u, v are Lie monomials of degree $< \mathscr{d}$. Applying the induction hypothesis to u and to v, we obtain

$$\alpha \in [I + L(Z), I + L(Z)] \subset [I, I] + [I, L(Z)] + [L(Z), L(Z)] \subset I + L(Z)$$

since I is a Lie ideal of $L(A)$. Therefore this ends our proof of (2) since it shows that $L(A)$ is the sum of $L(Z)$ and of I. Thus, we now prove that this sum is direct, that is to say that we have

$$I + L(Z) = I \oplus L(Z). \tag{3}$$

To show this result, it in fact suffices to prove that

$$I \cap L(Z) = \{0\}. \tag{4}$$

Thus let us consider the Lie algebra endomorphism λ of $L(X)$ defined on the letters of X by

$$\forall y \in Y \cup \bigcup_{i \in S} C(z_i),\ \lambda(y) = 0, \qquad \forall z \in Z,\ \lambda(z) = z$$
$$\forall i \in R - S,\ \forall x \in C(z_i) - \{z_i\},\ \lambda(x) = z_i.$$

Then it is clear that λ induces the identity on $L(Z)$ and the zero morphism on I according to Corollary II.2. The relation (4) follows now easily. Thus, we did show that $L(X)$ was a direct sum of I and of $L(Z)$. ∎

The following result follows immediately from the previous proposition:

COROLLARY II.4. *Let us take the notations of the previous proposition. Then we have the following isomorphism of the K-module*:

$$L(X)/I \simeq L(Z)$$

Note. This shows that we can take any basis of $L(Z)$ and in particular the Lyndon basis in order to obtain a basis for $L(X)/I$.

(2) *A Partially Commutative Version of Lazard's Elimination Theorem*

In this central section of our paper, we prove that the underlying K-module to every free partially commutative Lie algebra is a direct sum of free Lie algebras. But let us begin with some notations.

Up to now, B has been a fixed non empty subalphabet of a given alphabet A. Lazard's elimination theorem allows us to write

$$L(A) = L(B) \oplus L(T), \qquad (\mathscr{E}\mathscr{L})$$

where T is a basic subset generating the ideal $L(T)$ of $L(A)$ and defined by

$$T = \{\text{ad } w.z,\ w \in B^*,\ z \in A - B\}.$$

Let us introduce now the two ideals I_B and I_T of $L(B)$ and of $L(T)$:

$$I_B = I_\theta \cap L(B) \qquad \text{and} \qquad I_T = I_\theta \cap L(T).$$

Before giving our main result that makes precise the structure of the ideals I_B and I_T, we prove the following lemma:

LEMMA II.5. *Let α be a Lie monomial of $L(A)$. Then, we have*

$$\alpha \in L(B) \qquad \textit{or} \qquad \alpha \in L(T).$$

Proof. We establish this lemma by induction on the degree $\mathscr{d} \in \mathbb{N}^*$ of an homogeneous Lie element α. At first, observe that if $\mathscr{d} = 1$, we have

$$\alpha \in B \subset L(B) \qquad \text{or} \qquad \alpha \in A - B \subset T \subset L(T).$$

Thus this shows our result when $\mathscr{d} = 1$. Now let $\mathscr{d}$ be an integer $\geqslant 2$ and let us suppose that our result is proved at every order $< \mathscr{d}$. Then we can write

$$\alpha = [u, v],$$

where u and v are homogeneous Lie monomials of $L(A)$ of degree $< \mathscr{d}$. Hence we obtain by our induction hypothesis applied to u and v

$$\alpha \in [L(B), L(B)] \qquad \text{or} \qquad \alpha \in [L(T), L(T)] \qquad \text{or} \qquad \alpha \in [L(B), L(T)].$$

In the two first cases, we have clearly $\alpha \in L(B)$ or $\alpha \in L(T)$, and in the last case, $\alpha \in L(T)$, since $L(T)$ is an ideal of $L(A)$. Thus this ends our proof. ∎

COROLLARY II.6. *With the previous notations, we have*

$$I_\theta = I_B \oplus I_T.$$

Proof. Indeed, observe that I_θ is generated as Lie ideal by Lie monomials. Thus it follows that the K-module I_θ has a generating system made up with Lie monomials. But every Lie monomial α of I_θ belongs either to $L(B)$, or to $L(T)$ according to the previous lemma. This implies obviously that

$$\alpha \in I_B \cup I_T \subset I_B + I_T \Rightarrow I_\theta \subset I_B + I_T.$$

The corollary follows now easily from this last inclusion since according to $(\mathscr{E}\mathscr{L})$ and to the definition of I_B and I_T, the sum of the K-modules $I_B + I_T$ is necessarily direct. Moreover, the converse inclusion is obvious. ∎

Before stating our main theorem, let us introduce some more notations:

$$T_1 = \{\text{ad } wb.z,\ w \in B^*,\ b \in B,\ z \in A - B,\ (b, z) \in \theta\}$$

$$T_2 = \{\text{ad } wbcv.z - \text{ad } wcbv.z,\ w, z \in B^*,\ b, c \in B,\ (b, c) \in \theta,\ z \in A - B\}$$

$$\mathscr{B} = \{[b, c],\ b, c \in B,\ (b, c) \in \theta\}.$$

We can now give our main result that deals with the precise structure of generating systems for the two ideals I_B and L_T:

THEOREM II.7. *Let A be an alphabet and θ a partial commutation relation on A. Let us consider a non empty subset B of A such that $A - B$ is a* totally non commutative *subset for θ. Then, we have with the previous notations*

$$I_B = \langle \mathscr{B} \rangle_{L(B)} \qquad \text{and} \qquad I_T = \langle T_1 \cup T_2 \rangle_{L(T)}.$$

Proof. We call Lie θ-monomial every Lie monomial α of $L(A)$ of the form

$$\alpha = [\ldots.[a, b]\ldots.] \qquad \text{with} \quad (a, b) \in \theta.$$

By construction, the ideal I_θ is generated as a K-module by the family of the Lie θ-monomials. But, according to Lemma II.5, every Lie monomial of $L(A)$ and hence every Lie θ-monomial belongs either to $L(B)$, or to $L(T)$. It follows that the ideal I_T (resp. I_B) is generated as a module by the Lie θ-monomials of $L(A)$ which are in I_T (resp. in I_B). This remark being done, we can begin to prove our theorem. First, let us establish the part concerning B:

LEMMA II.8. *With the previous notations, we have*

$$I_B = \langle \mathscr{B} \rangle_{L(B)}.$$

Proof. First, in order to simplify our proof, let us denote

$$D = \langle \mathscr{B} \rangle_{L(B)}.$$

Then note that it is clear that $\mathscr{B}$ is included into $L(B)$. Thus this implies the obvious inclusion: $D \subset I_B$. To prove the converse inclusion, it suffices according to the preliminary remark to show that every Lie θ-monomial α of $L(A)$ that is in I_B belongs to D: we establish this result by induction on the homogeneous degree $\mathscr{d}$ of the Lie θ-monomial α. First, note that for the minimal value $\mathscr{d} = 2$ of a Lie θ-monomial in I_B, α can be written as

$$\alpha = [b, c] \qquad \text{with} \quad (b, c) \in \theta.$$

Since α belongs to $L(B)$ that can be identified to the sub-Lie-K-algebra of $K\langle A \rangle$ generated by B (cf. [Bo.1]), we can easily obtain that $b, c \in B$. It follows immediately that α belongs to $\mathscr{B}$ and hence to D. Let us suppose now that $\mathscr{d} \geqslant 3$ and that our result is proved at every order $< \mathscr{d}$. Then we can write

$$\alpha = \pm [u, v],$$

where u and v are Lie monomials of degree $< \mathscr{d}$ and where it can be supposed for instance that v is a Lie θ-monomial. According to Lemma II.5 and since $L(T)$ is a Lie ideal of $L(A)$, u and v necessarily belong to $L(B)$. Thus $v \in I_B$ and hence $v \in D$ according to the induction hypothesis. This implies that

$$\alpha \in [L(B), D] \subset D.$$

This ends our induction and proves the lemma. ∎

Before beginning to prove the second part of the theorem, let us introduce the notation that will simplify our proof:

$$H = \langle T_1 \cup T_2 \rangle_{L(T)}.$$

We now prove some intermediate lemmas:

LEMMA II.9. *With the previous notations, we have*

$$[L(B), H] \subset H.$$

Proof. We show by induction on the degree $\mathscr{d}$ of a Lie monomial α of $L(B)$ that we have the result

$$[\alpha, H] \subset H$$

that will prove our lemma. At first, observe that if $\mathscr{d} = 1$, we have

$$\alpha = b \in B \Rightarrow [\alpha, H] = \mathrm{ad}\, b.H \subset H,$$

since by construction H is obviously ad(b)-stable for every letter b of B. Let us consider now $\mathscr{d} \geqslant 2$ and let us suppose that our result is true at every order $< \mathscr{d}$. Then we can write

$$\alpha = [u, v],$$

where u and v are Lie monomials of degree $< \mathscr{d}$ in $L(B)$. By Jacobi's identity and from the induction hypothesis applied to u and v, we can write

$$[\alpha, H] = [[u, v], H] \subset [u, [v, H]] + [v, [u, H]] \subset [u, H] + [v, H] \subset H.$$

This ends our induction and proves the lemma. ∎

Remark. Since H is by definition an ideal of $L(T)$, Lemma II.9 and the relation ($\mathscr{E}\mathscr{L}$) imply immediately that H is an ideal of $L(A)$.

LEMMA II.10. *Let us consider b and c two letters of B such that $(b, c) \in \theta$. Then we have with the previous notations*

$$[L(T), [b, c]] \subset H.$$

Proof. To show this lemma, we prove by induction on the degree $\mathscr{d}$ of a Lie monomial α of $L(T)$ that under the hypotheses of the lemma, we have

$$[\alpha, [b, c]] \in H. \tag{1}$$

First, observe that when $\mathscr{d} = 1$, α is a letter of T. Hence, considered as an element of $L(A)$, it has the form

$$\alpha = \mathrm{ad}\, w.z \qquad \text{with} \quad w \in B^* \text{ and } z \in A - B.$$

Therefore, it follows by Jacobi's identity that

$$\begin{aligned}[\alpha, [b, c]] &= [\mathrm{ad}\, w.z, [b, c]] = [c, [b, \mathrm{ad}\, w.z]] - [b, [c, \mathrm{ad}\, w.z]] \\ &= \mathrm{ad}\, cbw.z - \mathrm{ad}\, bcw.z \in T_2 \subset H.\end{aligned}$$

Thus this ends by proving (1) for $\mathscr{d}=1$. Let us consider now $\mathscr{d} \geqslant 2$ and let us suppose that (1) is established at every order $<\mathscr{d}$. Then we can write

$$\alpha = [u, v],$$

where u and v are Lie monomials of $L(T)$ of degree $<\mathscr{d}$. Then applying Jacobi's identity, we obtain

$$[\alpha, [b, c]] = [[u, v], [b, c]] = [u, [v, [b, c]]] - [v, [u, [b, c]]].$$

Thus applying the induction hypothesis to u and v, we deduce

$$[\alpha, [b, c]] \in [u, H] + [v, H] \subset [L(T), H] \subset H.$$

This ends our induction and proves our lemma. ▮

LEMMA II.11. *With the previous notations, we have*

$$[L(T), I_B] \subset H.$$

Proof. We denote here by Lie θ-monomial of $L(B)$ every Lie monomial α of $L(B)$ that has the following form:

$$\alpha = [\ldots.[b, c]\ldots.] \qquad \text{with} \quad (b, c) \in \theta \text{ and } b, c \in B.$$

According to Lemma II.8, it will suffice to prove by induction on the degree $\mathscr{d}$ of a Lie θ-monomial α of $L(B)$ that we have

$$[L(T), \alpha] \subset H \tag{1}$$

in order to obtain our lemma. First, observe that when $\mathscr{d}$ has its minimal value that is 2 according to the Lemma II.8, we can immediately conclude with Lemma II.10. Let us suppose now that $\mathscr{d} \geqslant 2$ and that (1) is proved at every order $>\mathscr{d}$. Then we can write α in the form

$$\alpha = \pm [u, v],$$

where u and v are two Lie monomials of $L(B)$ and where we can suppose that v is a Lie θ-monomial of $L(B)$. Hence, using Jacobi's identity, we can write

$$[L(T), \alpha] = [L(T), [u, v]] \subset [u, [L(T), v]] + [v, [L(T), u]].$$

Hence, applying the induction hypothesis to v, by Lemma II.9 and since $L(T)$ is an ideal of $L(A)$, we obtain

$$[L(T), \alpha] \subset [u, H] + [v, [L(T), L(B)]] \subset [L(B), H] + [v, L(T)] \subset H.$$

This ends our induction and proves the lemma. ▮

We can now come to the proof of the second claim of our theorem. Thus, we shall show that we have

$$H = I_T. \tag{1}$$

First, observe that $H \subset I_T$ since this can easily be seen by looking at the proof of Lemma II.10 that showed that $T_2 \subset I_\theta$. Thus it will suffice to prove the converse inclusion. Therefore we show by induction on the degree $\mathscr{d}$ of a Lie θ-monomial α of $L(A)$ that is in $L(T)$ that we have

$$\alpha \in H. \tag{TH}$$

According to the initial remarks, this suffices to prove (1). First note that if $\mathscr{d} = 2$, which is the minimal possible value, we have

$$\alpha = [b, c] \qquad \text{with} \quad (b, c) \in \theta.$$

Since $A - B$ is totally non commutative for θ, the letters b and c cannot both belong to $A - B$. Thus, taking $-\alpha$ if necessary, we can suppose that $b \in B$. In this case, if c was belonging to B, we would have $\alpha \in L(B)$ which is not possible according to $(\mathscr{E}\mathscr{L})$. It follows that $c \in A - B$ and hence that $\alpha = \operatorname{ad} b . c \in T_1 \subset H$. Let us now consider $\mathscr{d} \geqslant 3$ and suppose that (TH) is proved at every order $< \mathscr{d}$. Then we can write

$$\alpha = \pm [u, v],$$

where u, v are Lie monomials of $L(A)$ of degree $< \mathscr{d}$ and where we can suppose that v is a Lie θ-monomial. Then several cases can occur:

— if $v \in L(T)$, then according to the induction hypothesis, we have $v \in H$. Since H is an ideal of $L(A)$ according to the remark following Lemma II.9, we can immediately conclude that $\alpha \in H$.

— if $v \in L(B)$, then according to Lemma II.5 and to the fact that $L(T)$ is an ideal of $L(A)$, we have $u \in L(T)$. Then, we have $\alpha \in [L(T), I_B]$ and hence $\alpha \in H$ by Lemma II.11.

Thus, we have shown that $\alpha \in H$ in every case. This ends our induction proving (1) and then our theorem. ▮

We now give another more explicit version of our theorem. For that purpose, let us introduce the definition:

DEFINITION II.1. Let A be an alphabet and let θ be a partial commutation relation on A. For every word w of A^*, we call *final alphabet of* w the subalphabet of A denoted $\operatorname{FAlph}(w)$ and defined by

$$\operatorname{FAlph}(w) = \{b \in A,\ \exists u \in A^*,\ w \equiv_\theta ub\}.$$

Remark. The reader will easily verify that we have for every $w_1, w_2 \in A^*$

$$w_1 \equiv_\theta w_2 \Rightarrow \mathrm{FAlph}(w_1) = \mathrm{FAlph}(w_2).$$

Thus we also denote $\mathrm{FAlph}(w)$ for $w \in M(A, \theta)$. By definition, this is the common value of the $\mathrm{FAlph}(v)$ for $\bar{v} = w$.

Let us also denote by S a section of the commutation classes for θ.[2] For every word w of A^*, we denote by $\mathscr{s}(w)$ the unique equivalent word for θ to w in S. Then let us consider the family

$$\mathscr{T} = \{\mathrm{ad}\, w.z,\ w \in B^* \cap S,\ z \in A - B,\ \forall b \in \mathrm{FAlph}(w),\ (b, z) \notin \theta\}.$$

We can now give the following corollary of Theorem II.7:

COROLLARY II.12. *Let us take again the previous notations. Then, the subset* $\mathscr{T}$ *is basic and generates a free Lie subalgebra* $L(\mathscr{T})$ *of* $L(T)$. *Moreover, the K-module* $L(T)$ *has the following direct sum decomposition*:

$$L(T) = I_T \oplus L(\mathscr{T}).$$

Proof. According to Theorem II.7, the Lie ideal I_T of $L(T)$ is generated by letters of T and by differences of letters of T. Thus we are in the framework of part (1) of this section. With Theorem II.7 and the definition of $\equiv_\theta$ (see [Ch]), it can be easily checked that the equivalence relation $\equiv$ on T used in II.1 can be defined here by

$$\forall w, v \in B^*,\ \forall z, z' \in A - B, \qquad \mathrm{ad}\, w.z \equiv \mathrm{ad}\, v.z' \Leftrightarrow z = z' \text{ and } w \equiv_\theta v.$$

Hence, if $\mathscr{S}$ denotes a section of the commutation classes for θ, the set

$$R = \{\mathrm{ad}\, w.z,\ w \in B^* \cap \mathscr{S},\ z \in A - B\}$$

is a section of the letters of T for $\equiv$. Then it is easy to see that if we set as in II.1, taking acount of Theorem II.7,

$$S = \{t \in R,\ \exists u \equiv t,\ u \in \mathscr{T}\},$$

we have the following identity:

$$S = \{\mathrm{ad}\, w.z,\ w \in B^* \cap \mathscr{S},\ z \in A - B,\ \exists b \in \mathrm{FAlph}(w),\ (b, z) \in \theta\}.$$

Since here every letter of T is equivalent modulo $\equiv$ to a letter of R, it follows easily from Proposition II.3 that the family $R - S$, which is exactly

[2] We can for instance use the Cartier–Foata normal forms (cf. [CaFo]) or the lexicographic normal forms (cf. [Ch, Pe]) to obtain easily such a system.

equal to $\mathcal{T}$, is a basic family generating a Lie subalgebra of $L(T)$ which is a supplementary of I_T in $L(T)$. Our corollary follows. ▮

Remark. We can also obtain this corollary in a different but equivalent way. Indeed let us introduce the families

$$\mathcal{T}_1 = \{\text{ad } w.z,\ w \in B^* \cap S,\ z \in A - B,\ \exists b \in \text{FAlph}(w),\ (b, z) \in \theta\}$$

$$\mathcal{T}_2 = \{\text{ad } w.z - \text{ad } \mathcal{s}(w).z,\ w \in B^*,\ w \neq \mathcal{s}(w),\ z \in A - B\}$$

$$\mathcal{T}_3 = \{\text{ad } w.z,\ w \in B^* \cap S,\ z \in A - B,\ \forall b \in \text{FAlph}(w),\ (b, z) \notin \theta\}.$$

Then it can be shown that the ideal I_T admits the generating family

$$I_T = \langle \mathcal{T}_1 \cup \mathcal{T}_2 \rangle_{L(T)}.$$

Then since the free K-modules generated by the family $\mathcal{T}' = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ and the family T are isomorphic, we can easily prove that the family $\mathcal{T}'$ is a basic family that generates $L(T)$:

$$L(T) = L(\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3).$$

Then our corollary follows in the same way from the preliminary study made in II.1. But we use here only the case where Dif $= \varnothing$ which is simpler.

Before giving the next result, we extend the operation "ad" to the free partially commutative Lie algebra. To this end, observe at first that we have, as can easily be checked,

$$w_1 \equiv_\theta w_2 \Rightarrow \forall z \in A,\ \text{ad } w_1.z - \text{ad } w_2.z \in I_\theta.$$

Thus, this being done, we can now give the definition:

DEFINITION II.2. Let $w \in M(A, \theta)$ and let $z \in A$. Then we denote ad $w.z$ the common value in $L(A, \theta)$ of the $\text{pl}_\theta(\text{ad } v.z)$ for $\bar{v} = w$.

We can now deduce from the previous theorem a partially commutative version of Lazard's elimination theorem:

THEOREM II.13. *Let A be an alphabet and let θ be a partial commutation relation on A. Let us consider a non empty subset B of A such that $A - B$ is a* totally non commutative *part for θ. Then let us denote by θ' the subgraph of θ induced by B and let us denote by $M(B, \theta')$ the submonoid of $M(A, \theta)$ generated by B. Let us introduce finally the family of $L(A, \theta)$:*

$$\mathcal{T} = \{\text{ad } w.z,\ w \in M(B, \theta'),\ z \in A - B,\ \forall b \in \text{FAlph}(w),\ (b, z) \notin \theta\}.$$

Then, the Lie subalgebra of $L(A, \theta)$ generated by $\mathcal{T}$ is isomorphic to a free Lie algebra with $\mathcal{T}$ as basic family that we will denote $L(\mathcal{T})$ and that is an ideal of $L(A, \theta)$. Moreover the K-module $L(A, \theta)$ admits the direct sum decomposition

$$L(A, \theta) = L(B, \theta') \oplus L(\mathcal{T}).$$

Proof. This is an immediate consequence of Corollaries II.6 and II.12, of Theorem II.7, and of Corollary II.4. ∎

Remarks. (1) Since $L(\mathcal{T})$ is a Lie ideal of $L(A, \theta)$, the decomposition of the previous theorem is in fact a semi-direct product where $L(B, \theta')$ acts by derivations on $L(\mathcal{T})$ (see [Bo.1, Bo.2] or [Lz.2]).

(2) Observe that the restriction to a totally non commutative part $A - B$ in the partially commutative elimination theorem is very important. Indeed, it implies that it is not possible to coarsely adapt to $L(A, \theta)$ the construction of the Lazard's bases of $L(A)$ (see [Vi.2] or [Lo]) which was based on the case $|B| = 1$ in the classical elimination theorem.

(3) *The K-module $L(A, \theta)$ Is Free*

Thus we deduce the following corollary on the basis of which constructions for $L(A, \theta)$ could now rely

COROLLARY II.14. *Let θ be a partial commutation relation on a given* finite *alphabet A. Then there exist a finite number of basic subsets $(T_i)_{i=1,N}$ in $L(A)$, all made with Lie monomials of $L(A)$, such that the K-module $L(A, \theta)$ admits the following decomposition in direct sum*:

$$L(A, \theta) = \bigoplus_{i=1}^{N} L(T_i). \qquad (\mathcal{DC}\theta)$$

Proof. This follows immediately from the previous theorem by an immediate induction on $|A|$ or on $|\theta|$ for instance. ∎

Remarks. (1) The reader will easily verify that the underlying algorithm to the proof of Corollary II.14 is the slowest on the cliques of θ. More precisely, if the above decomposition of $L(A, \theta)$ is constructed from Theorem II.13, it can be checked that N is greater or equal to the order of the greatest clique of the graph of θ.

(2) Corollary II.14 leads us naturally to introduce the notion of *free index* of a free partially commutative Lie algebra,

$$\operatorname{ind}(\theta) = \min \left\{ m, \exists (T_i)_{i=1,m} \in \mathcal{P}(L(A))^m, L(A, \theta) = \bigoplus_{i=1}^{m} L(T_i) \right\},$$

which is related in an intrinsic way to θ. It would be interesting to know if there exists an explicit expression of $\mathrm{ind}(\theta)$ in connection with θ. For instance, the reader can verify that $\mathrm{ind}(\theta)=1$ iff $\theta=\emptyset$.

(3) In fact the previous corollary can be easily adapted to the case of an infinite alphabet by the use of a transfinite induction.

Then we can easily show with the previous corollary that the homogeneous components of $L(A,\theta)$ are free. This was already known from [Du.2].

COROLLARY II.15. *Let A be an alphabet and let θ be a partial commutation relation on A. Then, we have*

(1) $\forall n \geqslant 1$, *$L_n(A,\theta)$ is a free K-module.*

(2) $\forall \alpha \in \mathbb{N}^{(A)}$, *$L_\alpha(A,\theta)$ is a free K-module.*

Proof. When the alphabet A is finite, this result comes from the fact that a free Lie algebra is a free K-module (cf. [Bo.2]) and from Corollary II.14 since the free Lie algebras $L(T_i)$ occuring there are graded for the two considered graduations since they are generated by Lie monomials.

In order to treat the case when A is an infinite alphabet, observe first that the Lie subalgebra of $L(A,\theta)$ generated by a subset $B \subset A$ can be identified by Corollary II.6 with $L(B,\theta')$, where θ' denotes the restriction of θ to B. With this identification, we can write for every $\alpha \in \mathbb{N}^{(A)}$

$$L_\alpha(A,\theta) = L_\alpha(\mathrm{Alph}(\alpha), \theta'_\alpha)$$

with obvious notations. Since the alphabet of α is finite, we conclude the K-module $L_\alpha(A,\theta)$ is free by the previous case. Then it is elementary to deduce from this that the K-module $L_n(A,\theta)$ is free for every $n \geqslant 0$. ∎

Then we obtain immediately that the K-module $L(A,\theta)$ is free:

COROLLARY II.16. *Let A be an alphabet and let θ be a partial commutation relation. Then, $L(A,\theta)$ is a free K-module.*

Proof. This is an obvious consequence of the previous corollary. ∎

Remarks. (1) More explicitly, Corollary II.14 permits us also to construct bases for $L(A,\theta)$. Indeed, Theorem II.13 allows the explicit construction of the families $(T_i)_{i=1,N}$ that occur in $(\mathscr{D}\mathscr{C}\theta)$. Hence, to obtain a K-basis for $L(A,\theta)$, it suffices to use any classical method (see [Vi.2]) to construct bases $(\mathscr{B}_i)_{i=1,N}$ for the free Lie algebras $(L(T_i))_{i=1,N}$: their union will obviously be a basis of $L(A,\theta)$.

(2) Since the families $(T_i)_{i=1,N}$ given by Corollary II.14 are made of Lie elements which are homogeneous for the multidegree, they also permit

us to construct easily bases for the homogeneous components of $L(A, \theta)$ for the total degree and for the multidegree.

(3) According to the Poincaré–Birkoff–Witt theorem, we also obtain again the fact that $L(A, \theta)$ embeds into $K\langle A, \theta\rangle$: this was already established in [Du.2].

EXAMPLES. (1) We give here a generic method of graph "disassembling" that is suited to our partially commutative elimination theorem. For this purpose, we suppose that A is fully ordered by $<$. Let us consider the alphabet $\mathrm{Alph}(\theta) = \{t_1, ..., t_N\}$ of θ ordered by $<$ such that

$$t_1 < t_2 < \cdots < t_N.$$

Then let us introduce the following subsets of $\mathrm{Alph}(\theta)$:

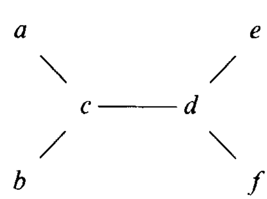
$$\forall i \in [\![1, N]\!], \qquad \theta(t_i) = \{t, (t_i, t) \in \theta, t_i < t\}.$$

Let us consider now the subalphabet $B \subset \mathrm{Alph}(\theta)$ formed of the letters b such that $\theta(b)$ is not empty. Then we denote

$$B = \{b_1, ..., b_t\} \qquad \text{with} \quad b_1 < b_2 < \cdots < b_t.$$

It is easy to see that $A - B$ is a totally non commutative part. This allows us to apply the elimination theorem in order to obtain a decomposition of $L(A, \theta)$ convenient for the computation of a K-basis.

(2) As an example, we use the above method to study the case where $A = \{a, b, c, d, e, f\}$ and where the graph of θ is the following:



The reader will verify that according to the order put on the letters of A, we can for instance obtain the parts $B = \{c, d\}$ or $B = \{a, b, c, f, c\}$ of A with the method described in (1). We deal here with the first case. Then Theorem II.13 allows us to write

$$L(A, \theta) = L(\mathcal{T}) \oplus L(c, d, \theta') = L(\mathcal{T}) \oplus K.c \oplus K.d,$$

where $\mathcal{T}$ is a basic family defined by

$$\mathcal{T} = \{\mathrm{ad}\, d^n.a, \mathrm{ad}\, d^n.b, \mathrm{ad}\, c^n.e, \mathrm{ad}\, c^n.f, n \geqslant 0\}.$$

Let us order this family by increasing homogeneous degree:

$$a < b < e < f < \operatorname{ad} c.e < \operatorname{ad} c.f < \operatorname{ad} d.a < \operatorname{ad} d.b < \cdots$$

Using the classical computation technique of the Lyndon basis for $L(\mathscr{T})$, which can be adapted without difficulties when the alphabet is infinite, we obtain for instance the homogeneous elements of degree 1 and 2 in $L(A, \theta)$,

$$L_1(A, \theta) = K.a \oplus K.b \oplus K.c \oplus K.d \oplus K.e \oplus K.f$$

$$\begin{aligned} L_2(A, \theta) = {} & K.[a, b] \oplus K.[a, e] \oplus K.[a, f] \oplus K.[b, e] \oplus K.[b, f] \\ & \oplus K.[e, f] \oplus K.[c, e] \oplus K.[c, f] \oplus K.[d, a] \oplus K.[d, b], \end{aligned}$$

which are here constructed by the computation of the Lyndon basis of $L(\mathscr{T})$.

(3) We also give here the first elements of a basis of $L(A, \theta)$ in the case where $A = \{a, b, c\}$ and where the commutation graph of θ is given by

$$a \longrightarrow b \qquad c$$

By use of Theorem II.13, we have

$$L(A, \theta) = L(a, \theta') \oplus L(\mathscr{T}) = K.a \oplus L(\mathscr{T}),$$

where we can take the family $\mathscr{T}$ of the form

$$\mathscr{T} = \{\operatorname{ad} a^n.c,\ n \geqslant 0\} \cup \{b\}.$$

Then we can consider the Lyndon basis of $L(\mathscr{T})$ associated to the natural order on $\mathscr{T}$ defined as follows:

$$b < c < \operatorname{ad} a.c < \operatorname{ad} a^2.c < \cdots < \operatorname{ad} a^n.c < \operatorname{ad} a^{n+1}.c < \cdots$$

In this case, we deduce easily a basis $\mathscr{L}y_\theta$ for the K-module $L(A, \theta)$; its first elements are given below:

$$\mathscr{L}y_\theta \cap L_1(A, \theta) = \{a, b, c\} \qquad \text{and} \qquad \mathscr{L}y_\theta \cap L_2(A, \theta) = \{[b, c], [a, c]\}$$

$$\begin{aligned} \mathscr{L}y_\theta \cap L_3(A, \theta) = {} & \{[a, [a, c]], [b, [a, c]], [c, [a, c]], [[b, c], c], \\ & [b, [b, c]]\} \end{aligned}$$

$$\begin{aligned} \mathscr{L}y_\theta \cap L_4(A, \theta) = {} & \{[a, [a, [a, c]]], [b, [a, [a, c]]], [c, [a, [a, c]]], \\ & [[b, c], [a, c]], [[b, [a, c]], c], [b, [b, [a, c]]], \\ & [c, [c, [a, c]]], [b, [b, [b, c]]], [b, [[b, c], c]], \\ & [[[b, c], c], c]\}. \end{aligned}$$

They are computed by the Lyndon method applied to $L(\mathscr{T})$.

III. Ranks of the Homogeneous Components of $L(A, \theta)$

(1) *Counting the Partially Commutative Words*

At first, we interest ourselves in the counting of the partially commutative words of given total degree n and multidegree α. Indeed, though this step is independent from the sequel, it is necessary to compute the ranks of the homogeneous components of $L(A, \theta)$. At first, before beginning the counting, we associate to θ the *commutative* series in the variables $(X_a)_{a \in A}$ (which is a polynomial when A is finite)

$$C(\theta) = \sum_{\substack{\alpha \in \{0,1\}^{(A)} \\ \mathrm{Alph}(\alpha) \in \mathscr{C}\ell(\theta)}} (-1)^{|\alpha|-1} X^{\alpha} \in \mathbb{Z}[[X_a]],$$

where X^{α} denotes classically for every $\alpha \in \mathbb{N}^{(A)}$ the monomial (cf. [Bo.2])

$$X^{\alpha} = \prod_{a \in \mathrm{supp}(\alpha)} X_a^{\alpha(a)}.$$

Then we define the family $(c(\alpha))_{\alpha \in \mathbb{N}^{(A)}}$ of rational numbers with the series

$$-\log(1 - C(\theta)) = \sum_{\alpha \in \mathbb{N}^{(A)}} c(\alpha) X^{\alpha}. \qquad (\ell g)$$

When the alphabet A is *finite*, we can define the $\mathbb{Q}$-algebra morphism v from $\mathbb{Q}[[X_a]]_{a \in A}$ into the $\mathbb{Q}$-algebra $\mathbb{Q}[[X]]$ of the formal series in one variable X by extension of the mapping which is defined by $v(X_a) = X$ for every a in A. Such a morphism does exist since A is finite. In this case, we can consider the polynomial of $\mathbb{Z}[X]$ defined by

$$C_t(\theta) = v(C(\theta)) \in \mathbb{Z}[X].$$

The reader will easily verify that we have

$$C_t(\theta) = \sum_{n \in \mathbb{N}} (-1)^{n-1} |\mathscr{C}\ell_n(\theta)| . X^n \in \mathbb{Z}[X],$$

where $\mathscr{C}\ell_n(\theta)$ is the set of the cliques of cardinality n in the commutation graph of θ. Then we can define the family $(c(m))_{m \geqslant 0}$ of $\mathbb{Q}^{\mathbb{N}}$ by

$$-\log(1 - C_t(\theta)) = \sum_{m \geqslant 0} c(m) X^m. \qquad (\ell g t)$$

These notations being given, the following proposition gives now the number of partially commutative words of given length or multidegree:

PROPOSITION III.1. *We respectively define for every multidegree $\alpha \in \mathbb{N}^{(A)}$ when the alphabet A is arbitrary and for every integer $n \geqslant 0$ when the alphabet is* finite, *the following positive integers*:

$$\forall \alpha \in \mathbb{N}^{(A)},\ m_\alpha = |M_\alpha(A, \theta)| \qquad \textit{and} \qquad \forall n \geqslant 0,\ m_n = |M_n(A, \theta)|.$$

Then these integers have the following generating series:

$$\sum_{\alpha \in \mathbb{N}^{(A)}} m_\alpha X^\alpha = [1 - C(\theta)]^{-1} \qquad (\mathcal{DM})$$

and

$$\sum_{n \in \mathbb{N}} m_n X^n = [1 - C_t(\theta)]^{-1}. \qquad (\mathcal{DMT})$$

Proof. At first we treat the case of multidegrees. Thus let us consider the generating series of the integers $(m_\alpha)_{\alpha \in \mathbb{N}^{(A)}}$:

$$S = \sum_{\alpha \in \mathbb{N}^{(A)}} m_\alpha X^\alpha.$$

Let us introduce now the Möbius function of $M(A, \theta)$ (see [Rt], [La], [CaFo], or [Ch]). It satisfies by definition the relation

$$\left(\sum_{w \in M(A,\theta)} w\right)\left(\sum_{w \in M(A,\theta)} \mu(w) w\right) = 1 \qquad (\mathcal{FM})$$

in $\mathbb{Z}\langle\langle A, \theta\rangle\rangle$. Observe now that the mapping $\tau: a \to X_a$ defines an algebra morphism from $\mathbb{Z}\langle\langle A, \theta\rangle\rangle$ in $\mathbb{Z}[[X_a]]_{a \in A}$ since it is ciear that every monomial of $\mathbb{Z}[[X_a]]_{a \in A}$ has only a finite number of converse images by τ. Note that

$$\tau\left(\sum_{w \in M(A,\theta)} w\right) = S \qquad \text{and} \qquad \tau\left(\sum_{w \in M(A,\theta)} \mu(w).w\right) = 1 - C(\theta).$$

Indeed, the first equality comes from the definition of S and the second from the definition of $C(\theta)$ according to the explicit value of the Möbius function of $M(A, \theta)$ (cf. [La], [CaFo], or [Ch]). Applying τ to the equation $(\mathcal{FM})$, we obtain the relation $S.[1 - C(\theta)] = 1$ and then $(\mathcal{DM})$. To obtain the formula $(\mathcal{DMT})$, it suffices now to apply ν to $(\mathcal{DM})$ since the image by ν of S is the generating series of the integers $(m_n)_{n \geqslant 0}$. ∎

EXAMPLE. We take again here the example given in II.3 of the alphabet $A = \{a, b, c\}$ equipped of $\theta = \{(a, b), (b, a)\}$. Then we have

$$C(\theta) = X_a + X_b + X_c - X_a X_b.$$

It follows immediately that

$$(1-C(\theta))^{-1}=\sum_{n\geqslant 0}\left(\sum_{p+q=n}\binom{n}{p}(-1)^q(X_a+X_b+X_c)^p(X_aX_b)^q\right). \quad (1)$$

According to III.1, this relation allows us to compute the numbers $m(\alpha)$ for α in $\mathbb{N}^{(A)}$. Let us show now for instance how to obtain $m(1,2,1)$. Remark that the total degree of the monomial $(X_a+X_b+X_c)^p(X_aX_b)^q$ is $p+2q$. Thus the monomial $X_aX_b^2X_c$ can only appear in (1) for pairs (p,q) such that $p+2q=4$, i.e., for

$$(p,q)\in\{(4,0),(2,1),(0,2)\}.$$

Then an elementary computation permits us to find the coefficient of $X_aX_b^2X_c$ in (1), which is equal to $m(1,2,1)$ according to $(\mathscr{DM})$:

$$m(1,2,1)=12.\binom{4}{0}-2.\binom{3}{1}+0.\binom{2}{0}=6.$$

(2) *Homogeneous Components of $L(A,\theta)$ for the Multidegree*

We count here the ranks of the homogeneous components for the multidegree of $K\langle A,\theta\rangle$ and $L(A,\theta)$. First, observe that it is clear from the definition that, for every $\alpha\in\mathbb{N}^{(A)}$, the K-module $K_\alpha\langle A,\theta\rangle$ is free of rank given by

$$m_\alpha=\mathrm{rg}_K\,K_\alpha\langle A,\theta\rangle.^3$$

Thus, since m_α is the number of words in $M(A,\theta)$ of multidegree α, the rank of $K_\alpha\langle A,\theta\rangle$ is obviously independent from K. Then, according to this remark, we can count the ranks of $L(A,\theta)$ for the multidegree:

THEOREM III.2. *For every $\alpha\in\mathbb{N}^{(A)}$, the K-module $L_\alpha(A,\theta)$ is free. Thus we can give the following definition:*

$$\forall\alpha\in\mathbb{N}^{(A)},\qquad \ell(\alpha)=\mathrm{rg}_K\,L_\alpha(A,\theta).$$

Then, for every $\alpha\in\mathbb{N}^{(A)}$, the rank $\ell(\alpha)$ is finite and is independent from K. Moreover, it is given by the formula

$$\ell(\alpha)=\sum_{d\mid\alpha}\mu(d)\frac{c(\alpha/d)}{d}. \qquad (\mathscr{DL})$$

[3] If K is a commutative ring and if L is a free K-module, we denote by $\mathrm{rg}_K\,L$ the rank of this free K-module.

Proof. First, note that for every $\alpha \in \mathbb{N}^{(A)}$, the K-module $L_\alpha(A, \theta)$ is free according to Corollary II.15. In the same way, the independence from K of the rank of $L_\alpha(A, \theta)$ comes also from the proof of II.15. Thus we can now suppose that $K = \mathbb{Q}$ in order to show $(\mathscr{D}\mathscr{L})$ since this will not affect our counting.

In order to prove the formula $(\mathscr{D}\mathscr{L})$, we adapt the argument of [Bo.2] that corresponded to the case $\theta = \varnothing$. First, let us choose now a $\mathbb{Q}$-basis $(a_{(i,\alpha)})_{1 \leqslant i \leqslant \ell(\alpha)}$ of $L_\alpha(A, \theta)$ for every α in $\mathbb{N}^{(A)} - \{0\}$. Then let us consider I the set of all pairs (i, α) such that $\alpha \in \mathbb{N}^{(A)} - \{0\}$ and $1 \leqslant i \leqslant \ell(\alpha)$. We may order I by an arbitrary full order $\leqslant_I$. According to the Poincaré–Birkoff–Witt Theorem (see [Bo.1]) and to Corollary I.2, the elements

$$P_\nu = \prod_{(i,\alpha) \in I} a_{(i,\alpha)}^{\nu(i,\alpha)}$$

for $\nu = (\nu(i, \alpha))$ running through $\mathbb{N}^{(I)}$ form a basis of $\mathbb{Q}\langle A, \theta\rangle$. Each P_ν is a homogeneous element of multidegree $\delta(\nu) = \alpha . |\nu|$. Then it follows that

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}^{(A)}} m_\alpha X^\alpha &= \sum_{\alpha \in \mathbb{N}^{(A)}} \dim_{\mathbb{Q}} \mathbb{Q}_\alpha\langle A, \theta\rangle X^\alpha = \sum_{\nu \in \mathbb{N}^{(I)}} X^{\delta(\nu)} = \sum_{\nu \in \mathbb{N}^{(I)}} X^{\alpha . |\nu|} \\ &= \sum_{\nu \in \mathbb{N}^{(I)}} \prod_{(i,\alpha) \in I} (X^\alpha)^{\nu(i,\alpha)} = \prod_{(i,\alpha) \in I} \left(\sum_{n \geqslant 0} (X^\alpha)^n \right) \\ &= \prod_{(i,\alpha) \in I} (1 - X^\alpha)^{-1} = \prod_{\alpha \neq 0} (1 - X^\alpha)^{-\ell(\alpha)}. \end{aligned}$$

Hence, according to the proof of Proposition III.1, we deduce that

$$[1 - C(\theta)]^{-1} = \prod_{\alpha \neq 0} (1 - X^\alpha)^{-\ell(\alpha)}. \qquad (\mathscr{D}\theta)$$

Taking the logarithm in $\mathbb{Q}[[X_a]]_{a \in A}$ of the two members of $(\mathscr{D}\theta)$, it follows that

$$\begin{aligned} -\log(1 - C(\theta)) &= -\log \prod_{\alpha \neq 0} (1 - X^\alpha)^{\ell(\alpha)} = -\sum_{\alpha \neq 0} \ell(\alpha) \log(1 - X^\alpha) \\ &= \sum_{\alpha \neq 0} \ell(\alpha) \sum_{m \geqslant 1} \frac{1}{m} X^{m\alpha} = \sum_{\alpha \neq 0} \left(\sum_{m\beta = \alpha} \ell(\beta) \frac{1}{m} \right) X^\alpha. \end{aligned}$$

Thus, according to (ℓg), we obtain the relation:

$$\sum_{\alpha \neq 0} \left(\sum_{m\beta = \alpha} \ell(\beta) \frac{1}{m} \right) X^\alpha = -\log(1 - C(\theta)) = \sum_{\alpha \neq 0} c(\alpha) X^\alpha$$

Then we can immediately deduce from this that we have

$$\forall \alpha \in \mathbb{N}^{(A)} - \{0\}, \qquad c(\alpha) = \sum_{m\beta = \alpha} \frac{1}{m} \ell(\beta). \tag{1}$$

Let us denote by n the gcd of the elements of the support of α. Then let us set $\alpha = n.\alpha_1$. Then the formula (1) can be rewritten in the form

$$c(n.\alpha_1) = \sum_{m \mid n} \frac{1}{m} \ell(\alpha/m) \Leftrightarrow n.c(n.\alpha_1) = \sum_{m \mid n} \frac{n}{m} \ell(\alpha_1.n/m)$$

that we can also write in the equivalent form

$$n.c(n.\alpha_1) = \sum_{m \mid n} m\ell(\alpha_1.m). \tag{2}$$

Applying the classical Möbius inversion formula in $\mathbb{N}^*$ to (2), considered as a relation between mappings of n and m, we obtain

$$n.\ell(\alpha_1.n) = \sum_{d \mid n} \mu(d).\frac{n}{d} c(\alpha_1.n/d).$$

Therefore we finally have

$$\ell(\alpha) = \sum_{d \mid \alpha} \mu(d) \frac{c(\alpha/d)}{d}. \tag{$\mathscr{DL}$}$$

This ends the proof of our theorem. ∎

EXAMPLES. (1) *Non commutative case.* If $\theta = \varnothing$, we have clearly

$$C(\theta) = \sum_{a \in A} X_a \Rightarrow -\log(1 - C(\theta)) = \sum_{n \geqslant 1} \frac{1}{n} \left(\sum_{a \in A} X_a \right)^n.$$

Applying the multinomial formula, we easily obtain

$$\forall \alpha \in \mathbb{N}^{(A)} - \{0\}, \qquad c(\alpha) = \frac{|\alpha|!}{\alpha!\, |\alpha|}.$$

Thus this gives us again the classical Witt's formula (cf. [Bo.2], [Wi]):

$$\ell(\alpha) = \frac{1}{|\alpha|} \sum_{d \mid \alpha} \mu(d) \frac{|\alpha/d|!}{(\alpha/d)!}.$$

(2) *Commutative case.* If $\theta = A^2 - \Delta_A$, we have easily

$$C(\theta) = 1 - \prod_{a \in A} (1 - X_a) \Rightarrow -\log(1 - C(\theta)) = \sum_{a \in A} \sum_{n \geqslant 1} \frac{1}{n} X_a^n.$$

It follows immediately that we have

$$c(\alpha) = \begin{cases} 1/n & \text{if } \alpha = n\varepsilon_a \\ 0 & \text{if not.} \end{cases}$$

Using the formula $(\mathscr{D}\mathscr{L})$ and the properties of the usual Möbius function, we easily obtain that $\ell(\varepsilon_a) = 1$ for every $a \in A$ and that $\ell(\alpha) = 0$ instead. Thus we find again that $\mathscr{L}(A, \theta) \simeq K^{(A)}$.

(3) For a really partially commutative case, let us take again the example given in II.3 and in III.1 of $A = \{a, b, c\}$ equipped with $\theta = \{(a, b), (b, a)\}$. According to the expression of $C(\theta)$ seen in III.1, we easily have

$$-\log(1 - C(\theta)) = \sum_{n \geqslant 0} \frac{1}{n} \left(\sum_{p+q=n} \binom{n}{p} (-1)^q (X_a + X_b + X_c)^p (X_a X_b)^q \right). \quad (*)$$

Thus, with the formula $(\mathscr{D}\mathscr{L})$ of Theorem III.2 and with a computation similar to the one made in III.1, we have

$$\ell(1, 2, 1) = c(1, 2, 1) = \frac{1}{2} 12. \binom{4}{0} - \frac{1}{3} 2. \binom{3}{1} + \frac{1}{2} 0. \binom{2}{0} = 1$$

since $(1, 2, 1)$ can only be divided by 1. Thus, we find again a result that can also be verified with the basis of $L_4(A, \theta)$ given in II.3.

(3) *Homogeneous Components of $L(A, \theta)$ for the Total Degree*

We shall count here the dimensions of the homogeneous components of $L(A, \theta)$ for the total degree. Thus we are obliged to suppose that the alphabet A is *finite*. First, remark that the K-module $K_n\langle A, \theta \rangle$ is obviously free for any integer $n \geqslant 0$ of rank

$$\forall n \geqslant 0, \qquad \mathrm{rg}_K\, K_n\langle A, \theta \rangle = m_n$$

which is independent of K. The same result holds also for $L_n(A, \theta)$:

THEOREM III.3. *Let A be a* finite *alphabet. Then the K-module $L_n(A, \theta)$ is free for every $n \in \mathbb{N}$. Thus we can define*

$$\forall n \in \mathbb{N}, \qquad \ell(n) = \mathrm{rg}_K\, L_n(A, \theta).$$

Then, for every $n \in \mathbb{N}$, *the* rank $\ell(n)$ *is finite and is independent of* K. *Moreover, it is given by the formula*

$$\ell(n) = \sum_{m \mid n} \mu(m) \frac{c(n/m)}{m}. \qquad (\mathscr{D}\mathscr{L}\mathscr{T})$$

Proof. First, the fact that $L_n(A, \theta)$ is free of rank independent from K comes immediately from Corollary II.15 and from its proof. To prove the formula $(\mathscr{D}\mathscr{L}\mathscr{T})$, let us transform at first the formula $(\mathscr{D}\theta)$ given in the proof of Theorem III.2 by the morphism ν introduced in III.1. Then we obtain

$$1 - C_t(\theta) = \prod_{\alpha \neq 0} (1 - X^{|\alpha|})^{\ell(\alpha)} = \prod_{m > 0} (1 - X^m)^{\ell(m)}. \qquad (1)$$

Taking the logarithm in $\mathbb{Q}[[X]]$ of the two members of (1), it follows that

$$\begin{aligned} -\log(1 - C_t(\theta)) &= -\log \prod_{m \neq 0} (1 - X^m)^{\ell(m)} \\ &= -\sum_{m \neq 0} \ell(m).\log(1 - X^m) = \sum_{m \neq 0} \ell(m).\left(\sum_{p \geqslant 1} \frac{1}{p} X^{pm}\right). \end{aligned}$$

Thus we deduce the relation

$$\sum_{n \neq 0} \left(\sum_{pm = n} \ell(m) \frac{1}{p}\right) X^n = -\log(1 - C_t(\theta)) = \sum_{n \neq 0} c(n).X^n.$$

Then we have for every $n \in \mathbb{N} - \{0\}$

$$c(n) = \sum_{pm = n} \ell(m) \frac{1}{p} \Rightarrow n.c(n) = \sum_{pm = n} m.\ell(m) = \sum_{m \mid n} m.\ell(m).$$

Then the Möbius inversion formula in $\mathbb{N}^*$ gives easily

$$\ell(n) = \sum_{m \mid n} \mu(m) \frac{c(n/m)}{m}.$$

Therefore this ends the proof of our theorem. ∎

Example. Let us take again the example given in III.1 and in III.2: it is $A = \{a, b, c\}$ and $\theta = \{(a, b), (b, a)\}$. Then the formula $(*)$ of III.2 gives

$$-\log(1 - C_t(\theta)) = \sum_{n \geqslant 0} \frac{1}{n} \left(\sum_{p + q = n} \binom{n}{p} (-1)^q \, 3^p X^{p + 2q}\right). \qquad (*t)$$

It follows very easily that:

$$\forall n \geqslant 0, \qquad c(n) = \sum_{p+2q=n} \frac{1}{p+q} \binom{p+q}{p} (-1)^q 3^p$$

Then, applying formula $(\mathscr{DLT})$, we find

$$\ell(1) = c(1) = 3, \qquad \ell(2) = c(2) - c(1)/2 = \tfrac{7}{2} - \tfrac{3}{2} = 2$$
$$\ell(3) = c(3) - c(1)/3 = 6 - \tfrac{3}{3} = 5$$

and

$$\ell(4) = c(4) - c(2)/2 = \tfrac{47}{4} - \tfrac{7}{4} = 10.$$

We also verify these relations by direct counting from the bases of $L_1(A, \theta)$, of $L_2(A, \theta)$, of $L_3(A, \theta)$, and of $L_4(A, \theta)$ given in II.3.

COROLLARY III.4. *Let A and B be two* finite *alphabets respectively equipped with two partial commutation relations θ and θ'. Then, if the Lie algebras $L(A, \theta)$ and $L(B, \theta')$ are isomorphic, we have*

$$\forall n \geqslant 1, \qquad |\mathscr{C}\ell_n(\theta)| = |\mathscr{C}\ell_n(\theta')|.$$

Proof. Let us recall that the lower central series $(\mathscr{L}^n(A, \theta))_{n \geqslant 1}$ of $L(A, \theta)$ is inductively defined by relations

$$\mathscr{L}^1(A, \theta) = \mathscr{L}(A, \theta)$$

and

$$\forall n \geqslant 1, \qquad \mathscr{L}^{n+1}(A, \theta) = [\mathscr{L}(A, \theta), \mathscr{L}^n(A, \theta)],$$

where here $[\mathscr{L}(A, \theta), \mathscr{L}^n(A, \theta)]$ denotes the sub-K-module of $L(A, \theta)$ generated by the family of the elements $[u, v]$ for $u \in \mathscr{L}(A, \theta)$ and $v \in \mathscr{L}^n(A, \theta)$. Then we can easily prove that we have for every $n \geqslant 1$

$$L_n(A, \theta) \simeq \mathscr{L}^n(A, \theta)/\mathscr{L}^{n+1}(A, \theta) \Rightarrow \ell(n) = \mathrm{rg}_K\, \mathscr{L}^n(A, \theta)/\mathscr{L}^{n+1}(A, \theta).$$

Hence the integer $\ell(n) \in \mathbb{N}$ is invariant by Lie algebra isomorphisms for every $n \geqslant 1$. According to the formula (1) given in the proof of Theorem III.3, it follows immediately that $C_\ell(\theta) = C_\ell(\theta')$. According to the definition of these two polynomials of $\mathbb{Z}[X]$, the corollary follows. ∎

Remarks. (1) With Theorem III.3, we can construct commutation relations on a same alphabet A with non isomorphic associated Lie algebras, but such that these free partially commutative Lie algebras have all their homogeneous components for the total degree isomorphic as K-modules. For instance, we can take the alphabet $A = \{a, b, c, d\}$ and the two following graphs:

$$\theta = a \text{ --- } b \text{ --- } c \text{ --- } d \qquad \text{and} \qquad \theta' = \begin{tikzcd}[column sep=small, row sep=small] & & d \\ a \arrow[r, dash] & b \arrow[ur, dash] \arrow[dr, dash] & \\ & & c \end{tikzcd}$$

Since these two graphs have the same numbers of cliques at every order, it follows immediately from Theorem III.3 that

$$\forall n \geqslant 1, \qquad \mathrm{rg}_K\, L_n(A, \theta) = \mathrm{rg}_K\, L_n(A, \theta').$$

Therefore the two following free K-modules are always isomorphic:

$$\forall n \geqslant 1, \qquad L_n(A, \theta) \simeq L_n(A, \theta').$$

We now briefly sketch the proof of the fact that the two Lie algebras $L(A, \theta)$ and $L(A, \theta')$ are not isomorphic. If $L(A, \theta)$ and $L(A, \theta')$ were isomorphic Lie algebras, their associated graded algebras would also be isomorphic (see [Lz.1]).

$$\begin{aligned} \mathrm{Grad}(L(A, \theta)) &= \bigoplus_{n=1}^{+\infty} \mathscr{L}^n(A, \theta)/\mathscr{L}^{n+1}(A, \theta) \simeq \mathrm{Grad}(L(A, \theta') \\ &= \bigoplus_{n=1}^{+\infty} \mathscr{L}^n(A, \theta')/\mathscr{L}^{n+1}(A, \theta'), \end{aligned}$$

where we take the notations of Corollary III.4. But as the graded algebra of a free partially commutative Lie algebra is isomorphic to it, it can be here easily checked that

$$\begin{aligned} \forall x \in \mathscr{L}^1(A, \theta)/\mathscr{L}^2(A, \theta) - \{0\}, &\qquad \mathrm{rg}_K(\mathrm{ad}(x)) \geqslant 1 \\ \text{For } \bar{b} \in \mathscr{L}^1(A, \theta')/\mathscr{L}^2(A, \theta'), &\qquad \mathrm{rg}_K(\mathrm{ad}(\bar{b})) = 0. \end{aligned}$$

Hence the Lie algebras $L(A, \theta)$ and $L(A, \theta')$ are not isomorphic, but all their homogeneous components for the total degree are isomorphic as K-modules.

(2) We do not know if the non isomorphism of the commutation graphs suffices to imply the non isomorphism of the corresponding free partially commutative Lie algebras.

Note added in proof. The above problem (2) has actually been solved positively in [DK].

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